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USE OF AN INERTIA SPHERE TO DAMP THE ANGULAR MOTIONS OF SPINNING SPACE VEHICLES

By JERROLD H. SUDDATH

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By Jerrold H. Suddath

SUMMARY

A theoretical study was made of a device which might be used to damp the angular motions of spin-stabilized space vehicles with constant moments of inertia. The device was assumed to consist of a rate gyro, a servo control, and a rotor mounted in a single gimbal. The investigation was conducted by considering the general equations of motion of the vehicle-damper system and noting that simplification would result if the damper had a spherical inertia distribution. Such a distribution was assumed thereafter, and a control command was defined so that the gimbal angle would be proportional to the angular velocity of the vehicle about the gimbal axis. The resulting equations were linearized, and the Routh-Hurwitz criterion was applied to determine the conditions for stability. The study included two numerical examples showing possible applications of inertia-sphere rate dampers.

The general conditions for stability were found to be feasible for practical applications. A simplified stability criterion covers a large class of practical problems.

INTRODUCTION

Spinning satellites which experience disturbance torques may develop precessional and nutational motions which interfere with scientific experiments and/or crew comfort in the case of manned missions. Therefore, a device which could reduce or eliminate such motions would have a real, practical value in some space missions.

A system which could control the attitude of a spinning space vehicle is discussed in reference 1. The purpose of this study was to investigate analytically the properties of a device which would damp the angular motions of spinning space vehicles with constant moments of inertia. The assumed device consists of a spinning body, a rate gyro, and a servo control mounted in the space vehicle. The center of mass of the spinning body would be located on a principal vehicle axis, and mounted in a gimbal with the gimbal axis parallel to a principal vehicle axis normal to the spin axis. The rate gyro would sense vehicle angular rates about the gimbal axis and supply a control command to the servo control. The servo control would apply a torque to the gimbal, and the reaction torques would damp the angular motions of the vehicle.

The general equations of motion of a vehicle with such a device were considered, and it was noted that a great deal of simplification would result if the spinning device had a spherical inertia distribution. Such a distribution was assumed thereafter, and a servo control command was defined. The resulting equations of motion were linearized, and the Routh-Hurwitz stability criterion was applied to the characteristic equation of the system. The study included two numerical examples of possible applications of inertia-sphere rate dampers.

SYMBOLS

A,B,C,D	constants used in characteristic equation (defined by eqs. (32) to (35))
a,b,c	coefficients (defined by eqs. (43) to (45))
$\underset{\sim}{E}$	identity matrix
$\mathbf{H}$	angular-momentum vector, slug-ft^2/sec
$I=I^*+I_D$, slug-ft^2	
I^*	transverse moment of inertia of vehicle when $I_Y=I_Z$, slug-ft^2

$\underset{\sim}{I}$	moment-of-inertia matrix, slug-ft²
I_D	moment of inertia of damper when $I_x=I_y=I_z$, slug-ft²
I_X,I_Y,I_Z	moments of inertia of vehicle about principal vehicle X-, Y-, and Z-axes, respectively, slug-ft²
I_x,I_y,I_z	moments of inertia of damper about principal damper x-, y-, and z-axes, respectively, slug-ft²
$I_1=I_X+I_D$, slug-ft²	
i	imaginary number, $\sqrt{-1}$
i,j,k	unit vectors along principal X-, Y-, and Z-axes, respectively
K	control sensitivity, sec
L	Lagrangian function, $T-V$, ft-lb
P	period, sec
p,q,r	angular velocities about principal X-, Y-, and Z-axes, respectively, radians/sec
p_o	positive constant spin rate of vehicle about X-axis, radians/sec
Q	generalized force or moment vector
Q_X,Q_Y,Q_Z	rolling, pitching, and yawing moments, respectively, in principal vehicle-axis coordinate system, ft-lb
$Q_{\delta_x},Q_{\delta_z}$	external torque acting upon rotor and gimbal, respectively, ft-lb
Q_{ξ_i}	component of external torque along ξ_i-axis
S	rotor spin vector, radians/sec
$S=\dot{\delta}_x$, radians/sec	
s	Laplace transform variable, per sec
T	kinetic energy, ft-lb
t	time, sec
$t_{1/2}$	time to damp to one-half amplitude, sec
$\mathbf{u}_{\xi_i}$	unit base vector of five-dimensional space
V	potential energy, ft-lb
X,Y,Z	principal vehicle-axis coordinates
X_I,Y_I,Z_I	inertial-axis coordinates
x,y,z	principal damper-axis coordinates
$\mathbf{\Gamma}$	Lagrangian vector operator
$\underset{\sim}{\Delta}$	orthogonal matrix which transforms vectors from principal vehicle-axis coordinate system to the principal damper-axis coordinate system
δ_x	angle generated by spin of damper about damper x-axis, radians
δ_z	angle of deflection of damper gimbal measured about Z-axis, radians
ϕ,θ,ψ	Euler angles, radians
$\xi_i=\phi,\ \theta,\ \psi,\ \delta_x,\ \delta_z$	(for $i=1, 2, 3, 4, 5$, respectively)
$\boldsymbol{\omega}$	angular-velocity vector, radians/sec
Subscripts:	
D	damper
i	integer
o	initial value
V	vehicle

A bar over a symbol indicates the Laplace transformation. Vectors are denoted by boldface letters. Dots over symbols indicate differentiation with respect to time. A tilde below a symbol denotes a matrix. A primed vector or matrix indicates the transposed vector or matrix.

ANALYSIS

DESCRIPTION OF SYSTEM

Figure 1 represents a vehicle-damper configuration. The X-, Y-, and Z-axes are principal vehicle axes. The vehicle spins about the X-axis to provide basic gyroscopic stability. The damper consists of a single gimbal, mounted with the gimbal axis along the Z-axis, and a rotor mounted in the gimbal. When the gimbal angle is zero, the rotor (shown as a sphere in the figure) spins about the X-axis.

Figure 2 illustrates the detail of a gimbal displacement. In position (a), the gimbal displacement is zero and the rotor spins about the X-axis. In position (b), the gimbal has been rotated through the angle δ_z in the positive sense. The

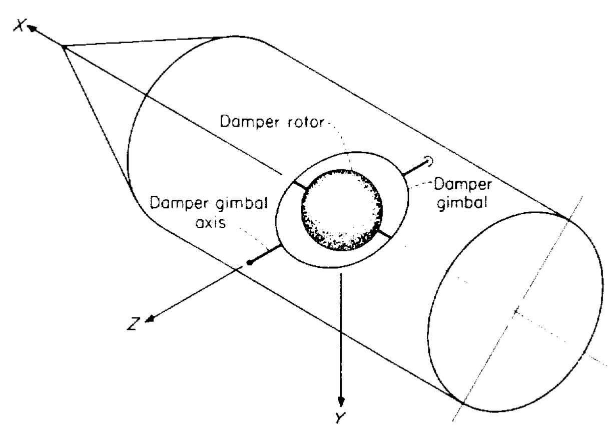

FIGURE 1. Illustration of vehicle-damper configuration. X, Y, and Z indicate the principal vehicle-fixed axes.

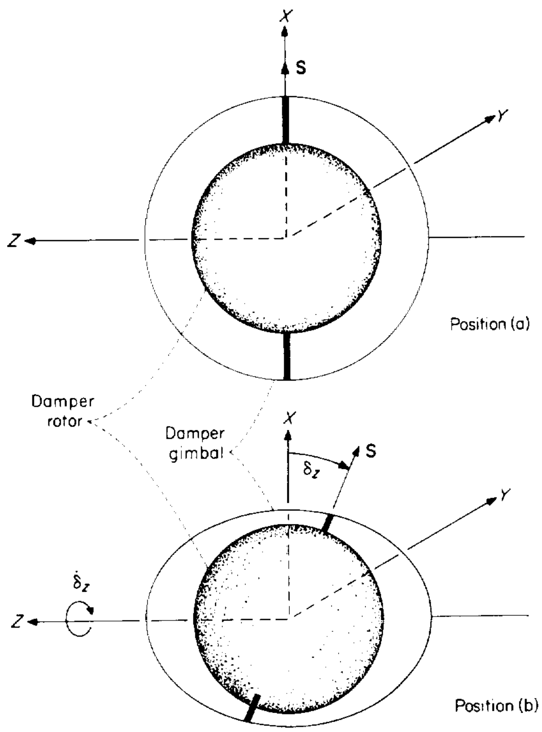

FIGURE 2.—Detail of gimbal displacement.

vector **S** is the spin vector of the rotor.

The function of the damper may be described qualitatively in the following way. Suppose that, initially, the gimbal is locked with $\delta_z \equiv 0$. Also suppose that the vehicle has experienced some disturbance and that the X-axis is not alined with the total-angular-momentum vector. From reference 2, it can be seen that in this condition, the vehicle would cone around the angular-momentum vector (which would be fixed in space) with a maximum angular deflection from a space-fixed reference which would be greater than the deflection of the angular-momentum vector from that axis. Since the total angular momentum of the vehicle plus damper must be constant (no external torques acting after the disturbance, for example), a change in the angular-momentum vector of the damper requires an equal and opposite change in the angular-momentum vector of the vehicle. The purpose of the damper in this case would be to eliminate the coning motion by alining the X-axis with the total-angular-momentum vector.

EQUATIONS OF MOTION

Basic equations.—The analysis is restricted to cases with no coupling from the force to the moment equations. The basic equations to be used are the five moment equations corresponding to five degrees of angular freedom of the vehicle-damper system. The coordinate systems used in the study are illustrated in figure 3. The five variables used in the Lagrangian formulation of the equations are necessarily ϕ, θ, ψ, δ_x, and δ_z. However, the Lagrangian and the final form of the equations will be in terms of δ_x, δ_z, and the angular rates about the principal vehicle axes, p, q, and r. A method for making the appropriate changes in variables is given in the appendix.

The following definitions are used to obtain the equations of motion:

$$\xi_i = \phi,\ \theta,\ \psi,\ \delta_x,\ \delta_z \quad \text{(for } i=1, 2, 3, 4, 5\text{, respectively)} \tag{1}$$

The Lagrangian vector operator $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma} = \sum_{i=1}^{5} \mathbf{u}_{\xi_i} \left[\frac{d}{dt} \left(\frac{\partial}{\partial \dot{\xi}_i} \right) - \frac{\partial}{\partial \xi_i} \right] \tag{2}$$

where $\mathbf{u}_{\xi_i}$ is a unit base vector of the five-dimensional space defined by the five degrees of freedom of the system. The Lagrangian function L is defined by

$$L = T - V \tag{3}$$

where T and V are the kinetic and potential energies of the system, respectively. The generalized force vector $\mathbf{Q}$ is defined by

$$\mathbf{Q} = \sum_{i=1}^{5} \mathbf{u}_{\xi_i} Q_{\xi_i} \tag{4}$$

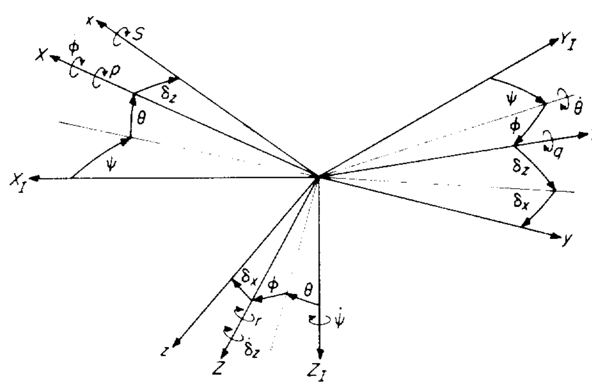

FIGURE 3.—Orientation of x, y, and z damper axes, and X, Y, and Z vehicle axes relative to X_I, Y_I, and Z_I inertial axes. The relationships are described by the Euler angles ϕ, θ, and ψ, the gimbal angle δ_z, and the damper spin angle δ_x.

where Q_{ξ_i} is the generalized force or moment corresponding to ξ_i. With these definitions, the equations of motion are obtained by setting

$$\Gamma L = \mathbf{Q} \tag{5}$$

For the present problem, V is taken to be zero. If the center of mass of the damper is located on a principal axis of the vehicle, the appropriate moment of inertia of the vehicle can be defined so as to include the damper as a point mass located at the damper center of mass. This case is the one considered herein. With these considerations, the Lagrangian function is given by

$$L = \frac{1}{2}\left(\boldsymbol{\omega}_V' \underset{\sim}{I}_V \boldsymbol{\omega}_V + \boldsymbol{\omega}_D' \underset{\sim}{I}_D \boldsymbol{\omega}_D\right) \tag{6}$$

where $\boldsymbol{\omega}_V'$ is the transpose of $\boldsymbol{\omega}_V$ which is the column angular-velocity vector of the vehicle-axis system. Similarly, $\boldsymbol{\omega}_D'$ is the transpose of $\boldsymbol{\omega}_D$ which is the column angular-velocity vector of the damper-axis system. The quantities $\underset{\sim}{I}_V$ and $\underset{\sim}{I}_D$ are the moment-of-inertia matrices of the vehicle and damper, respectively. The forms of $\boldsymbol{\omega}_V$ and $\underset{\sim}{I}_V$ are as follows:

$$\boldsymbol{\omega}_V = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{7}$$

and

$$\underset{\sim}{I}_V = \begin{bmatrix} I_X & 0 & 0 \\ 0 & I_Y & 0 \\ 0 & 0 & I_Z \end{bmatrix} \tag{8}$$

Let the notation $(\boldsymbol{\omega}_D)_V$ denote the fact that $\boldsymbol{\omega}_D$ is written in the vehicle-axis system. Then, it is easily seen that

$$(\boldsymbol{\omega}_D)_V = \begin{bmatrix} p + S\cos\delta_z \\ q + S\sin\delta_z \\ r + \dot{\delta}_z \end{bmatrix} \tag{9}$$

where $S \equiv \dot{\delta}_x$ is the x-component of the angular-velocity vector of the damper-axis system relative to the vehicle axes. In other words, if S and δ_z were identically zero, the inertial angular velocity of the damper axes would be the same as that of the vehicle.

In general, the moment-of-inertia matrix $\underset{\sim}{I}_D$ for arbitrary damper-rotor configurations is diagonal and constant only if it is determined relative to a set of principal damper axes. Therefore, the term $\boldsymbol{\omega}_D' \underset{\sim}{I}_D \boldsymbol{\omega}_D$ will be written relative to the damper-axis system. Let the notation $(\boldsymbol{\omega}_D)_D$ denote the fact that $\boldsymbol{\omega}_D$ is written in the damper-axis system; then,

$$(\boldsymbol{\omega}_D)_D = \underset{\sim}{\Delta}(\boldsymbol{\omega}_D)_V \tag{10}$$

where $\underset{\sim}{\Delta}$ is the orthogonal transformation matrix defined by

$$\underset{\sim}{\Delta} = \begin{bmatrix} \cos\delta_z & \sin\delta_z & 0 \\ -\sin\delta_z\cos\delta_x & \cos\delta_z\cos\delta_x & \sin\delta_x \\ \sin\delta_z\sin\delta_x & -\cos\delta_z\sin\delta_x & \cos\delta_x \end{bmatrix} \tag{11}$$

Finally, the second term in the Lagrangian function (eq. (6)) can be written as

$$\boldsymbol{\omega}_D' \underset{\sim}{I}_D \boldsymbol{\omega}_D = (\boldsymbol{\omega}_D)_V' \underset{\sim}{\Delta}' \underset{\sim}{I}_D \underset{\sim}{\Delta}(\boldsymbol{\omega}_D)_V \tag{12}$$

where $\underset{\sim}{I}_D$ is of the form

$$\underset{\sim}{I}_D = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \tag{13}$$

Thus, the complete five-component vector equation of this study can be written as

$$\Gamma\left\{\frac{1}{2}\left[\boldsymbol{\omega}_V' \underset{\sim}{I}_V \boldsymbol{\omega}_V + (\boldsymbol{\omega}_D)_V' \underset{\sim}{\Delta}' \underset{\sim}{I}_D \underset{\sim}{\Delta}(\boldsymbol{\omega}_D)_V\right]\right\} = \mathbf{Q} \tag{14}$$

General spherical damper equations.—Attention is now returned to the right-hand side of equation (12). In particular, the factor $\underset{\sim}{\Delta}' \underset{\sim}{I}_D \underset{\sim}{\Delta}$ is to be considered. Clearly, this factor is an orthogonal transformation of the matrix $\underset{\sim}{I}_D$ (see ref. 3), but more important is the fact that a spherical inertia distribution of the damper reduces this term to a scalar times the identity matrix $\underset{\sim}{E}$; thus, many terms are eliminated from the Lagrangian function.

In order to prove this statement, let $I_x = I_y = I_z = I_D$. Then

$$\underset{\sim}{I}_D = I_D \underset{\sim}{E} \tag{15}$$

and, since $\underset{\sim}{\Delta}$ is orthogonal,

$$\underset{\sim}{\Delta}' \underset{\sim}{I}_D \underset{\sim}{\Delta} = I_D \underset{\sim}{\Delta}' \underset{\sim}{E} \underset{\sim}{\Delta} = I_D \underset{\sim}{\Delta}' \underset{\sim}{\Delta} = I_D \underset{\sim}{E} \tag{16}$$

Thus, the proof is complete.

Hereinafter, the inertia distribution of the damper is taken to be spherical so that $\underset{\sim}{I_D}=I_D\underset{\sim}{E}$ and equation (14) is reduced to

$$\mathbf{\Gamma}\left\{\frac{1}{2}\left[\boldsymbol{\omega}_V'\underset{\sim}{I_V}\boldsymbol{\omega}_V+I_D(\boldsymbol{\omega}_D)_V'(\boldsymbol{\omega}_D)_V\right]\right\}=\mathbf{Q} \quad (17)$$

The analysis is restricted to vehicle configurations with

$$I_X\neq I_Y=I_Z=I^* \quad (18)$$

By going through the Lagrangian formulation with the change in variables discussed in the appendix, equation (17) may be written as the five following scalar equations:

$$I_1\dot{p}+I_D(\dot{S}\cos\delta_z-S\dot{\delta}_z\sin\delta_z)+I_Dq\dot{\delta}_z-I_DSr\sin\delta_z=Q_X \quad (19)$$

$$I\dot{q}+(I_1-I)pr+I_D(\dot{S}\sin\delta_z+S\dot{\delta}_z\cos\delta_z)+I_DSr\cos\delta_z-I_D\dot{\delta}_zp=Q_Y \quad (20)$$

$$I\dot{r}+(I-I_1)pq+I_D\ddot{\delta}_z+I_DS(p\sin\delta_z-q\cos\delta_z)=Q_Z \quad (21)$$

$$I_D\frac{d}{dt}(S+p\cos\delta_z+q\sin\delta_z)=Q_{\delta_x} \quad (22)$$

$$I_D\dot{r}+I_D\ddot{\delta}_z+I_DS(p\sin\delta_z-q\cos\delta_z)=Q_{\delta_z} \quad (23)$$

where

$$I_1=I_X+I_D \quad (24)$$

and

$$I=I^*+I_D \quad (25)$$

It should be noted that the left-hand side of equation (23) is contained in the left-hand side of equation (21). This is due to the fact that from familiar rigid-body dynamics,

$$I^*\dot{r}+(I^*-I_X)pq=Q_Z-Q_{\delta_z} \quad (26)$$

where Q_Z is the external torque acting on the vehicle, and $-Q_{\delta_z}$ is the reaction torque due to the damper. By taking Q_{δ_z} to the left-hand side in equation (26) and replacing it with the left-hand side of equation (23), equation (21) is obtained identically.

Linearized spherical damper equations.—The following assumptions are made in determining the linearized spherical damping equations:

I. $Q_X=Q_{\delta_x}=0$

II. The gimbal angle δ_z is always small enough to consider $\cos\delta_z=1$ and $\sin\delta_z=\delta_z$.

III. Terms containing the products $\dot{\delta}_z\delta_z$, $q\dot{\delta}_z$, $r\delta_z$, and $q\delta_z$ are small quantities and may be neglected.

IV. The spin rate of the vehicle p is constant and positive; that is, $\dot{p}=0$ and $p=p_o>0$.

V. The servo control is ideal in the sense that $\delta_z(t)$ will have whatever value is called for.

With assumptions I to IV, equations (19) and (22) simply give S=Constant. With assumption V, equation (23) simply gives the torque output of the servo control. Thus, the linear analysis is based on the following two equations:

$$I\dot{q}+[(I_1-I)p_o+I_DS]r=Q_Y+I_D(p_o-S)\dot{\delta}_z \quad (27)$$

$$I\dot{r}-[(I_1-I)p_o+I_DS]q=Q_Z-I_D\ddot{\delta}_z-I_DSp\delta_z \quad (28)$$

In order to determine a value of $\delta_z(t)$ which will provide damping, assume that the terms containing $\delta_z(t)$ and its derivatives provide damping, and then make the following considerations:

(a) The left-hand sides of equations (27) and (28) have the functional form of a vehicle with no damper.

(b) The damping moment in pitch is proportional to $\dot{\delta}_z$.

(c) If the $\ddot{\delta}_z$ term in equation (28) is small compared with the δ_z term, then the damping moment in yaw is proportional to δ_z.

(d) In reference 2, it was pointed out that damping can be introduced by a pitching moment proportional to $\dot{r}$ and a yawing moment proportional to r.

Therefore, it seems straightforward to choose

$$\delta_z=Kr \quad (29)$$

where the constant K will be referred to as the control sensitivity, or gain. With this choice of δ_z, equations (27) and (28) are rewritten as

$$\dot{q}+A\dot{r}+Br=\frac{Q_Y}{I} \quad (30)$$

$$-Bq+C\ddot{r}+\dot{r}+Dr=\frac{Q_Z}{I} \quad (31)$$

where

$$A=\frac{I_D(S-p_o)K}{I} \quad (32)$$

$$B=\frac{(I_1-I)p_o+I_DS}{I} \tag{33}$$

$$C=\frac{I_DK}{I} \tag{34}$$

$$D=\frac{I_DSp_oK}{I} \tag{35}$$

Taking the Laplace transformations of equations (30) and (31) gives

$$s\overline{q}+(As+B)\overline{r}=\frac{\overline{Q}_Y}{I}+q_o+Ar_o \tag{36}$$

$$-B\overline{q}+(Cs^2+s+D)\overline{r}=\frac{\overline{Q}_Z}{I}+(Cs+1)r_o+C\dot{r}_o \tag{37}$$

from which the cubic characteristic equation is

$$Cs^3+s^2+(AB+D)s+B^2=0 \tag{38}$$

SYSTEM STABILITY

Derivation of general system stability criterion.—The Routh-Hurwitz stability criterion (see ref. 4) states that all the roots of equation (38) will have negative real parts if the following conditions hold:

I. $C>0$
II. $B^2>0$
III. $AB+D-CB^2>0$

Condition I: From equation (34), $C>0$ holds only if $K>0$. Hereinafter K is taken to be positive.

Condition II: Since B is real, $B^2\geqq 0$; therefore, the case where $B=0$ must be avoided. It can be noted that B is zero when

$$S=\frac{I^*-I_X}{I_D}p_o \tag{39}$$

hence, this value of S must be avoided in the design of a stable system.

Condition III: For $B\neq 0$, the remaining condition which must be satisfied is given by

$$AB+D-CB^2>0 \tag{40}$$

By substituting the expressions for A, B, C, and D given in equations (32) to (35) into inequality (40), the following inequality is obtained:

$$I_D(I-I_D)S^2+(II_1+II_D-2I_1I_D)p_oS-I_1(I_1-I)p_o^2>0 \tag{41}$$

Now consider the left-hand side of inequality (41) as a quadratic function of S defined by

$$F(S)=aS^2+bS+c \tag{42}$$

where

$$a=I_D(I-I_D)=I_DI^* \tag{43}$$

$$b=(II_1+II_D-2I_1I_D)p_o \tag{44}$$

$$c=-I_1(I_1-I)p_o^2 \tag{45}$$

Since $a>0$, as $|S|\to\infty$, $F(S)\to+\infty$; thus, there are two cases to consider: (1) Either $F(S)>0$ for all real values of S or (2) there are two values of S, say $S^{(1)}$ and $S^{(2)}$ with $S^{(1)}\leqq S^{(2)}$, such that $S^{(1)}\leqq S\leqq S^{(2)}$ implies $F(S)\leqq 0$.

In the first case, $F(S)>0$ for all real values of S. If $F(S)>0$ for all real values of S, then solving $F(S)=0$ for S must give complex solutions, a fact which implies

$$b^2<4ac \tag{46}$$

By substituting expressions for a, b, and c given in equations (43), (44), and (45), the following inequality is obtained:

$$I_X^2<0 \tag{47}$$

which cannot be true. Therefore, $F(S)$ cannot be positive for all real values of S.

In the second case, $S^{(1)}\leqq S\leqq S^{(2)}$ implies $F(S)\leqq 0$. Solving $F(S)=0$ for $S^{(1)}$ and $S^{(2)}$ gives

$$S^{(1)}=-p_o\left(1+\frac{I_X}{I_D}\right) \tag{48}$$

$$S^{(2)}=p_o\left(\frac{I_X}{I^*}-1\right) \tag{49}$$

The general stability requirements are:

I. K is positive.
II. $S\neq\frac{I^*-I_X}{I_D}p_o$
III. Either $S<S^{(1)}$ or $S>S^{(2)}$.

Simplified stability criterion.—If S is restricted to positive values, the following conditions are sufficient for stability.

I. For vehicles with $I^* < I_X$ (disklike configuration),

$$S > p_o > 0 \tag{50}$$

implies stability. This condition follows from the fact that the sum of any two principal moments of inertia of a body must be greater than or equal to the third principal moment of inertia, so $I_X \leqq 2I^*$, and from equation (49) $S^{(2)} \leqq p_o$. Also, equation (39) does not hold since the right-hand side is negative.

II. For vehicles with $I_X = I^*$ (spherical configuration),

$$S > 0 \tag{51}$$

implies stability. This condition follows from the fact that $S^{(2)} = 0$, and the right-hand side of equation (39) is zero.

III. For vehicles with $I^* > I_X$ (pencillike configuration),

$$0 < S \neq \frac{I^* - I_X}{I_D} p_o \tag{52}$$

implies stability.

NUMERICAL EXAMPLES

Two cases were selected to illustrate applications of spherical dampers. The first case, a pencillike vehicle configuration, was taken to be representative of the spinning payloads of some state-of-the-art space vehicles. The second case, a toroidal vehicle configuration, was taken to represent the type of vehicle which might be used for a manned space station.

PENCILLIKE VEHICLE

The assumption $Q_Y = Q_Z = r_o = \dot{r}_o = 0$, equations (30) and (31), and the data given in table I were used to calculate the time histories of q/q_o and r/q_o plotted in figure 4. The damper rotor for the pencillike vehicle was assumed to be a spherical shell with a 6-inch radius and a weight of 2 pounds. If the spherical shell were made of a high-grade steel, the structural integrity of the shell should be adequate for the spin rates S and p_o used in this numerical example. The total weight of the damper system (excluding power supply) was estimated to be about 3.5 pounds whereas the vehicle weight (without damper) was considered to be about 350 pounds.

TABLE I.—VALUES OF PARAMETERS USED FOR NUMERICAL EXAMPLES

Parameter	Configuration	
	Pencillike	Toroidal
I, slug-ft²	40.21	389.207
I_1, slug-ft²	6.26	754.207
I_D, slug-ft²	0.01	207
K, sec	0.25	0.25
p_o, radians per sec	23.3	1.27
q_o, radians per sec	0.42	0.401
S, radians per sec	1,400	120

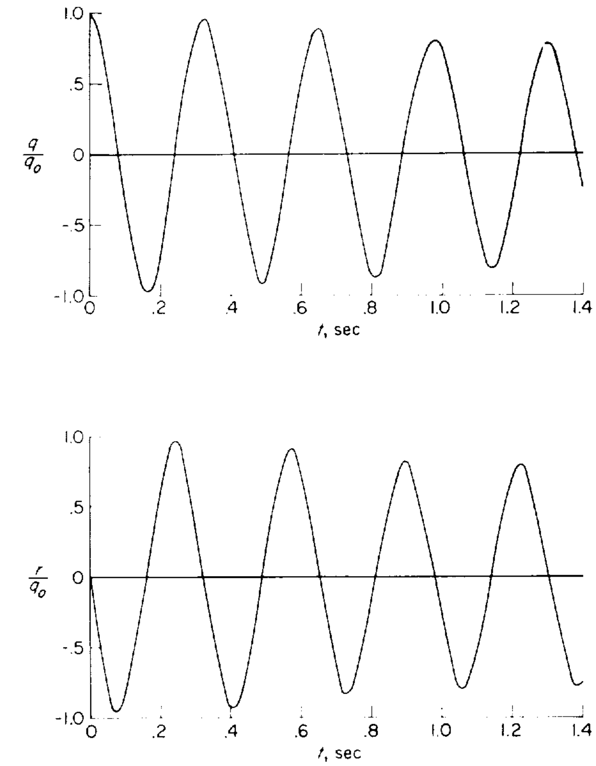

FIGURE 4. Time histories of q/q_o and r/q_o for numerical example of pencillike vehicle with spherical damper. $t_{1/2} = 3.95$ seconds; $P = 0.325$ second.

In this example, the real root had a large negative value so that q and r appear as damped oscillations with a time to damp to one-half amplitude of 3.95 seconds and a period of 0.325 second.

TOROIDAL VEHICLE

The assumption $Q_Y = Q_Z = r_o = \dot{r}_o = 0$, equations (30) and (31), and the data given in table I were used to calculate the time histories of q/q_o and r/q_o plotted in figure 5. The damper rotor, located at the center of the toroid, was assumed to be a high-grade-steel spherical shell with a radius of 6.58 feet and a weight of 223 pounds. The total weight of the damper system would be

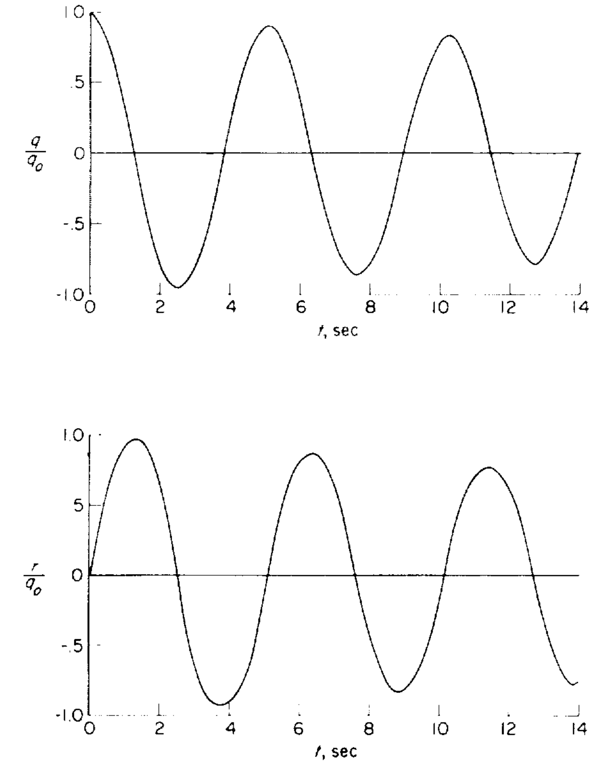

FIGURE 5.—Time histories of q/q_o and r/q_o for numerical example of toroidal vehicle with spherical damper. $t_{1/2}=34.76$ seconds; $P=5.1$ seconds.

300 to 350 pounds. The toroidal vehicle configuration was assumed to be generated by revolving a circle with a 5-foot radius about an axis 20 feet from its center. The total vehicle weight was considered to be 29 tons.

In this example, the real root had a large negative value so that q and r appear as damped oscillations with a time to damp to one-half amplitude of 34.76 seconds and a period of 5.1 seconds.

GENERAL DISCUSSION OF NUMERICAL EXAMPLES

Since both numerical examples of this study demonstrated a large separation between the real and oscillatory roots, it seemed reasonable to assume that there should be some simple method for estimating the roots of the characteristic equation (eq. (38)). This approximation was made in the following manner. Consider the cubic expression

$$(Cs+1)[s^2+(AB+D)s+B^2]$$
$$=Cs^3+[1+C(AB+D)]s^2$$
$$+[(AB+D)+CB^2]s+B^2 \quad (53)$$

If $|CB^2|<<|AB+D|$ and $|C(AB+D)|<<1$, then setting the left-hand side of equation (53) equal to zero gives a good approximation of the characteristic equation and a simple means of estimating the roots. For the two numerical cases of this study, the roots estimated in this manner are compared with the actual roots of equation (38) in table II, and they are seen to be in good agreement.

It should be noted that the term $(AB+D)$, which governs the damping of the oscillation in cases for which equation (53) can be used, can be written as

$$AB+D=KSp_o\frac{I_DI_1}{I^2}\left[\left(1-\frac{p_o}{S}\right)\left(1+\frac{I_DS}{I_1p_o}\right)+\frac{Ip_o}{I_1S}\right] \quad (54)$$

It is seen that the damping of the oscillation is proportional to the gain constant K. On the other hand, the real root is approximated by

$$-\frac{1}{C}=-\frac{I}{I_DK} \quad (55)$$

and is inversely proportional to K. Therefore, one might draw the rather obvious conclusion that for a given vehicle-damper system, there should be an optimum value of the gain. This facet of the problem is not treated herein. However, equations (54) and (55) indicate that for some practical applications, the system designer has a good degree of latitude in the selection of system performance and weight through the choice of values for K, S, and I_D.

TABLE II.—COMPARISON OF ROOTS OF CHARACTERISTIC EQUATION OBTAINED BY EXACT AND APPROXIMATE METHODS FOR NUMERICAL EXAMPLES

Roots	Configuration	
	Pencillike	Toroidal
Exact	-16084.58 $-0.17542\pm19.324i$	-7520.736 $-0.019935\pm1.2397i$
Approximate	-16084.93 $-0.18703\pm19.324i$	-7520.776 $-0.020037\pm1.2547i$

CONCLUDING REMARKS

A theoretical study was made of a device which might be used to damp the angular motions of spin-stabilized space vehicles. The device was

assumed to consist of a rate gyro, a servo control, and a single gimbal-mounted rotor. The basic moment equations for an axially symmetric vehicle with a spherical damper were derived and linearized. A control command signal was defined so that the gimbal deflection was made proportional to vehicle yaw rate, and the general conditions for stability were obtained. These conditions were found to be feasible for most problems of interest. The general stability criterion can be simplified and still cover a large class of practical applications.

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APPENDIX

LAGRANGIAN FORMULATION OF EQUATIONS WITH CHANGE OF VARIABLES

The problem of eliminating the Euler angles and their rates from the Lagrangian formulation of the moment equations is discussed in reference 5. However, reference 5 deals with a three-degree-of-freedom system whereas the system considered in this study has five degrees of rotational freedom. Therefore, this appendix is devoted to presenting some of the details of the formulation of the equations used in the study.

Consider first the Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=Q_{\phi} \tag{A1}$$

with the Euler angles defined as shown in figure 3. The expressions relating p, q, and r to the Euler angular rates (see ref. 6) are

$$p=\dot{\phi}-\dot{\psi}\sin\theta \tag{A2}$$

$$q=\dot{\theta}\cos\phi+\dot{\psi}\sin\phi\cos\theta \tag{A3}$$

$$r=\dot{\psi}\cos\phi\cos\theta-\dot{\theta}\sin\phi \tag{A4}$$

from which

$$\frac{\partial L}{\partial \dot{\phi}}=\frac{\partial L}{\partial p}\frac{\partial p}{\partial \dot{\phi}}+\frac{\partial L}{\partial q}\frac{\partial q}{\partial \dot{\phi}}+\frac{\partial L}{\partial r}\frac{\partial r}{\partial \dot{\phi}}=\frac{\partial L}{\partial p} \tag{A5}$$

It is easy to show that

$$\frac{\partial q}{\partial \phi}=r \tag{A6}$$

$$\frac{\partial r}{\partial \phi}=-q \tag{A7}$$

$$\frac{\partial p}{\partial \phi}=0 \tag{A8}$$

Thus,

$$\frac{\partial L}{\partial \phi}=\frac{\partial L}{\partial p}\frac{\partial p}{\partial \phi}+\frac{\partial L}{\partial q}\frac{\partial q}{\partial \phi}+\frac{\partial L}{\partial r}\frac{\partial r}{\partial \phi}=r\frac{\partial L}{\partial q}-q\frac{\partial L}{\partial r} \tag{A9}$$

Since $Q_{\phi}\equiv Q_X$, equation (A1) can be replaced by the equivalent expression

$$\frac{d}{dt}\left(\frac{\partial L}{dp}\right)+q\frac{\partial L}{\partial r}-r\frac{\partial L}{\partial q}=Q_X \tag{A10}$$

Note that by using the Lagrangian function written in terms of p, q, r, δ_x, $\dot{\delta}_x$, δ_z, and $\dot{\delta}_z$, equation (A10) is independent of the Euler angles. That is to say, equation (A10) involves only quantities which are measured relative to the XYZ system; therefore, it must be independent of the order in which the Euler rotations are taken. Thus equation (A10) will not be affected if the order of rotations is changed.

Now suppose that the order of the Euler angular rotations is defined as shown in figure 6(a). With this definition, the following relationships are true:

$$Q_{\theta}=Q_Y \tag{A11}$$

$$q=\dot{\theta}-\dot{\phi}\sin\psi \tag{A12}$$

$$r=\dot{\psi}\cos\theta+\dot{\phi}\cos\psi\sin\theta \tag{A13}$$

$$p=\dot{\phi}\cos\psi\cos\theta-\dot{\psi}\sin\theta \tag{A14}$$

$$\frac{\partial q}{\partial \dot{\theta}}=1 \tag{A15}$$

$$\frac{\partial p}{\partial \dot{\theta}}=\frac{\partial r}{\partial \dot{\theta}}=0 \tag{A16}$$

$$\frac{\partial q}{\partial \theta}=0 \tag{A17}$$

$$\frac{\partial p}{\partial \theta}=-r \tag{A18}$$

$$\frac{\partial r}{\partial \theta}=p \tag{A19}$$

Thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right)-\frac{\partial L}{\partial \theta}=Q_\theta \tag{A20}$$

can be replaced by the equivalent expression

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q}\right)+r\frac{\partial L}{\partial p}-p\frac{\partial L}{\partial r}=Q_Y \tag{A21}$$

By the same argument as used previously, equation (A21) is independent of the Euler angles and the order in which the rotations are taken.

Finally, if the order of rotations is defined as shown in figure 6(b), the same procedure used before would lead to the following equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial r}\right)+p\frac{\partial L}{\partial q}-q\frac{\partial L}{\partial p}=Q_Z \tag{A22}$$

Thus the equations of the study may be derived by writing the Lagrangian function in terms of p, q, r, δ_x, $\dot\delta_x$, δ_z, and $\dot\delta_z$ and using the following expressions:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial p}\right)+q\frac{\partial L}{\partial r}-r\frac{\partial L}{\partial q}=Q_X \tag{A23}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q}\right)+r\frac{\partial L}{\partial p}-p\frac{\partial L}{\partial r}=Q_Y \tag{A24}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial r}\right)+p\frac{\partial L}{\partial q}-q\frac{\partial L}{\partial p}=Q_Z \tag{A25}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\delta_x}\right)-\frac{\partial L}{\partial \delta_x}=Q_{\delta_x} \tag{A26}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\delta_z}\right)-\frac{\partial L}{\partial \delta_z}=Q_{\delta_z} \tag{A27}$$

It is of interest to note that if ∇_ω is defined so that

$$\nabla_\omega L\equiv\begin{bmatrix}\partial L/\partial p\\ \partial L/\partial q\\ \partial L/\partial r\end{bmatrix} \tag{A28}$$

then equations (A23), (A24), and (A25) can be written as

$$\frac{d}{dt}(\nabla_\omega L)+\boldsymbol{\omega}_V\times\nabla_\omega L=Q_X\mathbf{i}+Q_Y\mathbf{j}+Q_Z\mathbf{k} \tag{A29}$$

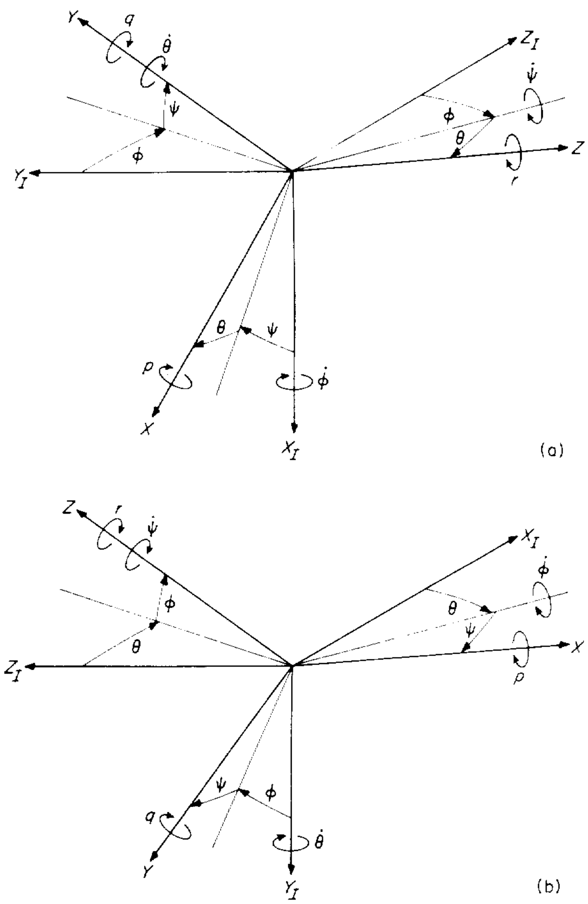
If L were simply the kinetic energy of a vehicle with no damper, given by

$$L=\frac{1}{2}\boldsymbol{\omega}_V'\underset{\sim}{I}_V\boldsymbol{\omega}_V \tag{A30}$$

then

$$\nabla_\omega L=\underset{\sim}{I}_V\boldsymbol{\omega}_V=\mathbf{H}_V \tag{A31}$$

and equation (A29) would be Euler's equations (see ref. 5) in vector form.



(a) ϕ, ψ, θ order.
(b) θ, ϕ, ψ order.

FIGURE 6.—Illustration of alternate choices of the order in which the Euler rotations may be taken.

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